

TAIL POSITIVE WORDS AND GENERALIZED COINVARIANT ALGEBRAS

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ABSTRACT. Let n, k , and r be nonnegative integers and let S_n be the symmetric group. We introduce a quotient $R_{n,k,r}$ of the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$ in n variables which carries the structure of a graded S_n -module. When $r \geq n$ or $k = 0$ the quotient $R_{n,k,r}$ reduces to the classical coinvariant algebra R_n attached to the symmetric group. Just as algebraic properties of R_n are controlled by combinatorial properties of permutations in S_n , the algebra of $R_{n,k,r}$ is controlled by the combinatorics of objects called *tail positive words*. We calculate the standard monomial basis of $R_{n,k,r}$ and its graded S_n -isomorphism type. We also view $R_{n,k,r}$ as a module over the 0-Hecke algebra $H_n(0)$, prove that $R_{n,k,r}$ is a projective 0-Hecke module, and calculate its quasisymmetric and nonsymmetric 0-Hecke characteristics. We conjecture a relationship between our quotient $R_{n,k,r}$ and the delta operators of the theory of Macdonald polynomials.

1. INTRODUCTION

Consider the action of the symmetric group S_n on n letters on the polynomial ring $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \dots, x_n]$ given by variable permutation. The polynomials belonging to the invariant subring

$$(1.1) \quad \mathbb{Q}[\mathbf{x}_n]^{S_n} := \{f(\mathbf{x}_n) \in \mathbb{Q}[\mathbf{x}_n] : \pi.f(\mathbf{x}_n) = f(\mathbf{x}_n) \text{ for all } \pi \in S_n\}$$

are the *symmetric polynomials* in the variable set \mathbf{x}_n . Let $e_d(\mathbf{x}_n)$ be the *elementary symmetric function* of degree d , that is $e_d(\mathbf{x}_n) = \sum_{1 \leq i_1 < \dots < i_d \leq n} x_{i_1} \cdots x_{i_d}$. It is well known that the set $\{e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n)\}$ gives an algebraically independent homogeneous collection of generators for the ring $\mathbb{Q}[\mathbf{x}_n]^{S_n}$.

Let $\mathbb{Q}[\mathbf{x}_n]_+^{S_n} \subset \mathbb{Q}[\mathbf{x}_n]^{S_n}$ be the subspace of symmetric polynomials with vanishing constant term. The *invariant ideal* $I_n \subseteq \mathbb{Q}[\mathbf{x}_n]$ is the ideal

$$(1.2) \quad I_n := \langle \mathbb{Q}[\mathbf{x}_n]_+^{S_n} \rangle = \langle e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n) \rangle$$

generated by this subspace. The *coinvariant algebra* R_n is the corresponding quotient:

$$(1.3) \quad R_n := \mathbb{Q}[\mathbf{x}_n] / I_n.$$

The algebra R_n is a graded S_n -module.

The coinvariant algebra is among the most important representations in algebraic combinatorics; algebraic properties of R_n are deeply tied to combinatorial properties of permutations in S_n . E. Artin proved [2] that the collection of ‘sub-staircase’ monomials $\{x_1^{i_1} \cdots x_n^{i_n} : 0 \leq i_j < j\}$ descends to a vector space basis for R_n , so that the Hilbert series of R_n is given by

$$(1.4) \quad \text{Hilb}(R_n; q) = (1+q)(1+q+q^2) \cdots (1+q+\dots+q^{n-1}) = [n]!_q,$$

the standard q -analog of $n!$. Chevalley [4] proved that as an *ungraded* S_n -module, we have $R_n \cong \mathbb{Q}[S_n]$, the regular representation of S_n . Lusztig (unpublished) and Stanley [16] refined this result to describe the *graded* isomorphism type of R_n in terms of the major index statistic on standard Young tableaux.

In this paper we will study the following generalization of the coinvariant algebra R_n . Recall that the degree d *homogeneous symmetric function* in $\mathbb{Q}[\mathbf{x}_n]$ is given by $h_d(\mathbf{x}_n) := \sum_{1 \leq i_1 \leq \dots \leq i_d \leq n} x_{i_1} \cdots x_{i_d}$.

Definition 1.1. Let n, k , and r be nonnegative integers with $r \leq n$. Let $I_{n,k,r} \subseteq \mathbb{Q}[\mathbf{x}_n]$ be the ideal

$$I_{n,k,r} := \langle h_{k+1}(\mathbf{x}_n), h_{k+2}(\mathbf{x}_n), \dots, h_{k+n}(\mathbf{x}_n), e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-r+1}(\mathbf{x}_n) \rangle$$

and let

$$R_{n,k,r} := \mathbb{Q}[\mathbf{x}_n] / I_{n,k,r}$$

be the corresponding quotient ring.

The ideal $I_{n,k,r}$ is homogeneous and stable under the action of the symmetric group, so that $R_{n,k,r}$ is a graded S_n -module. Since the generators of $I_{n,k,r}$ are symmetric polynomials, we have the containment of ideals $I_{n,k,r} \subseteq I_n$, so that $R_{n,k,r}$ projects onto the classical coinvariant algebra R_n . If $r \geq n$ or $k = 0$ we have the equality $I_{n,k,r} = I_n$, so that $R_{n,k,r} = R_n$.

Just as algebraic properties of R_n are controlled by combinatorics of permutations $\pi_1 \dots \pi_n$ of the set $\{1, 2, \dots, n\}$, algebraic properties of $R_{n,k,r}$ will be controlled by the combinatorics of permutations $\pi_1 \dots \pi_{n+k}$ of the multiset $\{0^k, 1, 2, \dots, n\}$ whose last r entries $\pi_{n+k-r+1} \dots \pi_{n+k-1} \pi_{n+k}$ are all nonzero. Thinking of positive letters as weights, we will call such permutations *r-tail positive*.

Let $S_{n,k,r}$ be the collection of all *r-tail positive* permutations of the multiset $\{0^k, 1, 2, \dots, n\}$. For example, we have

$$S_{2,2,1} = \{0012, 0021, 0102, 0201, 1002, 2001\}.$$

By considering the possible locations of the k 0's in an element of $S_{n,k,r}$, it is immediate that

$$(1.5) \quad |S_{n,k,r}| = \binom{n+k-r}{k} \cdot |S_n| = \binom{n+k-r}{k} \cdot n!.$$

The basic enumeration of Equation 1.5 will manifest in Hilbert series as

$$(1.6) \quad \text{Hilb}(R_{n,k,r}; q) = \left[\begin{matrix} n+k-r \\ k \end{matrix} \right]_q \cdot \text{Hilb}(R_n; q) = \left[\begin{matrix} n+k-r \\ k \end{matrix} \right]_q \cdot [n]!_q,$$

where $\left[\begin{matrix} m \\ i \end{matrix} \right]_q := \frac{[m]!_q}{[i]!_q [m-i]!_q}$ is the usual q -binomial coefficient. Going even further, we have the following graded Frobenius image

$$(1.7) \quad \text{grFrob}(R_{n,k,r}; q) = \left[\begin{matrix} n+k-r \\ k \end{matrix} \right]_q \cdot \text{grFrob}(R_n; q) = \left[\begin{matrix} n+k-r \\ k \end{matrix} \right]_q \cdot \sum_{T \in \text{SYT}(n)} s_{\text{shape}(T)},$$

which implies that the quotient $R_{n,k,r}$ consists of $\binom{n+k-r}{k}$ copies of the coinvariant algebra R_n , with grading shifts given by a q -binomial coefficient. The authors know of no direct way to see this from Definition 1.1.

The ideal $I_{n,k,r}$ defining the quotient $R_{n,k,r}$ is of ‘mixed’ type – its generators come in two flavors: the homogeneous symmetric functions $h_{k+1}(\mathbf{x}_n), h_{k+2}(\mathbf{x}_n), \dots, h_{k+n}(\mathbf{x}_n)$ and the elementary symmetric functions $e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-r+1}(\mathbf{x}_n)$. Several mixed ideals have recently been introduced to give combinatorial generalizations of the coinvariant algebra.

- Let $k \leq n$. Haglund, Rhoades, and Shimozono [10] studied the quotient of $\mathbb{Q}[\mathbf{x}_n]$ by the ideal

$$(1.8) \quad \langle x_1^k, x_2^k, \dots, x_n^k, e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n) \rangle.$$

The generators are high degree S_n -invariants $e_n(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n)$ together with a homogeneous system of parameters x_1^k, \dots, x_n^k of degree k carrying the defining representation of S_n . Algebraic properties of the corresponding quotient are controlled by combinatorial properties of k -block ordered set partitions of $\{1, 2, \dots, n\}$.

- Let $r \geq 2$ and let $\mathbb{Z}_r \wr S_n$ be the group of $n \times n$ monomial matrices whose nonzero entries are r^{th} complex roots of unity (this is the group of ‘ r -colored permutations’ of $\{1, 2, \dots, n\}$). Let $k \leq n$ be non-negative integers. Chan and Rhoades [3] studied the quotient of $\mathbb{C}[\mathbf{x}_n]$ by the ideal

$$(1.9) \quad \langle x_1^{kr+1}, x_2^{kr+1}, \dots, x_n^{kr+1}, e_n(\mathbf{x}_n^r), e_{n-1}(\mathbf{x}_n^r), \dots, e_{n-k+1}(\mathbf{x}_n^r) \rangle,$$

where $f(\mathbf{x}_n^r) = f(x_1^r, \dots, x_n^r)$ for any polynomial f . The generators here are high degree $\mathbb{Z}_r \wr S_n$ -invariants $e_n(\mathbf{x}_n^r), \dots, e_{n-k+1}(\mathbf{x}_n^r)$ together with a h.s.o.p. $x_1^{kr+1}, \dots, x_n^{kr+1}$ of degree $kr+1$ carrying the dual of the defining representation of $\mathbb{Z}_r \wr S_n$. Algebraic properties of the corresponding quotient are controlled by k -dimensional faces in the Coxeter complex attached to $\mathbb{Z}_r \wr S_n$.

- Let \mathbb{F} be any field and let $H_n(0)$ be the 0-Hecke algebra over \mathbb{F} of rank n ; the algebra $H_n(0)$ acts on the polynomial ring $\mathbb{F}[\mathbf{x}_n]$ by isobaric divided difference operators. Let $k \leq n$ be positive integers. Huang and Rhoades [13] studied the quotient of $\mathbb{F}[\mathbf{x}_n]$ by the ideal

$$(1.10) \quad \langle h_k(x_1), h_k(x_1, x_2), \dots, h_k(x_1, x_2, \dots, x_n), e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n) \rangle.$$

Once again, the generators consist of high degree $H_n(0)$ -invariants $e_n(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n)$ together with a h.s.o.p. of degree k carrying the defining representation of $H_n(0)$. Algebraic properties of the quotient are controlled by 0-Hecke combinatorics of k -block ordered set partitions of $\{1, \dots, n\}$.

The novelty of this paper is that our mixed ideals consist of high degree invariants of different kinds: elementary and homogeneous. It would be interesting to develop a more unified picture of the algebraic and combinatorial properties of mixed quotients of polynomial rings.

Our analysis of the rings $R_{n,k,r}$ will share many properties with the analyses of the previously mentioned mixed quotients. Since $I_{n,k,r}$ is not cut out by a regular sequence of homogeneous polynomials in $\mathbb{Q}[\mathbf{x}_n]$, the usual commutative algebra tools (e.g. the Koszul complex) used to study the coinvariant algebra R_n are unavailable to us. These will be replaced by *combinatorial* commutative algebra tools (e.g. Gröbner theory). We will see that the ideal $I_{n,k,r}$ has an explicit minimal Gröbner basis (with respect to the lexicographic term order) in terms of Demazure characters. This Gröbner basis will yield the Hilbert series of $R_{n,k,r}$, as well as an identification of its standard monomial basis. The graded S_n -isomorphism type of $R_{n,k,r}$ will then be obtainable by constructing an appropriate short exact sequence to serve as a recursion.

The rest of the paper is organized as follows. In **Section 2** we give background related to symmetric functions and Gröbner bases. In **Section 3** we determine the Hilbert series of $R_{n,k,r}$ and calculate the standard monomial basis for $R_{n,k,r}$ with respect to the lexicographic term order. In **Section 4** we determine the graded S_n -isomorphism type of $R_{n,k,r}$. We also view $R_{n,k,r}$ as a module over the 0-Hecke algebra $H_n(0)$ and calculate its graded nonsymmetric and bigraded quasisymmetric 0-Hecke characteristics. We close in **Section 5** with some open problems.

2. BACKGROUND

2.1. Words, partitions, and tableaux. Let $w = w_1 \dots w_n$ be a word in the alphabet of non-negative integers. An index $1 \leq i \leq n-1$ is a *descent* of w if $w_i > w_{i+1}$. The *descent set* of w is $\text{Des}(w) := \{1 \leq i \leq n-1 : w_i > w_{i+1}\}$ and the *major index* of w is $\text{maj}(w) := \sum_{i \in \text{Des}(w)} i$. A pair of indices $1 \leq i < j \leq n$ is called an *inversion* of w if $w_i > w_j$; the *inversion number* $\text{inv}(w)$ counts the inversions of w . The word w is called *r -tail positive* if its last r letters $w_{n-r+1} \dots w_n$ are positive.

Let $n \in \mathbb{Z}_{\geq 0}$. A *partition* λ of n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ of positive integers with $\lambda_1 + \dots + \lambda_k = n$. We write $\lambda \vdash n$ or $|\lambda| = n$ to indicate that λ is a partition of n . The *Ferrers diagram* of λ (in English notation) consists of λ_i left-justified boxes in row i . The Ferrers diagram of $(4, 2, 2) \vdash 8$ is shown below on the left.

If $V = \bigoplus_{d \geq 0} V_d$ is a graded vector space, the *Hilbert series* of V is the power series

$$(2.5) \quad \text{Hilb}(V; q) = \sum_{d \geq 0} \dim(V_d) \cdot q^d.$$

Similarly, if $V = \bigoplus_{d \geq 0} V_d$ is a graded S_n -module, the *graded Frobenius character* of V is

$$(2.6) \quad \text{grFrob}(V; q) = \sum_{d \geq 0} \text{Frob}(V_d) \cdot q^d.$$

2.3. Quasisymmetric and nonsymmetric functions. The space Λ of symmetric functions has many generalizations; in this paper we will also use the spaces QSym of quasisymmetric functions and \mathbf{NSym} of noncommutative symmetric functions. We briefly review their definition below, as well as their relationship with the 0-Hecke algebra $H_n(0)$; for more details see [12, 13].

Let n be a positive integer. A (strong) composition α of n is a sequence $(\alpha_1, \dots, \alpha_k)$ of positive integers with $\alpha_1 + \dots + \alpha_k = n$. We write $\alpha \models n$ or $|\alpha| = n$ to indicate that α is a composition of n . The map $\alpha = (\alpha_1, \dots, \alpha_k) \mapsto \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \dots + \alpha_{k-1}\}$ provides a bijection between compositions of n and subsets of $[n-1]$; we will find it convenient to identify compositions with subsets.

Let $S \subseteq [n-1]$ be a subset. The *Gessel fundamental quasisymmetric function* $F_S = F_{S,n}$ attached to S is the degree n formal power series

$$(2.7) \quad F_S = F_{S,n} := \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}.$$

The space QSym of *quasisymmetric functions* is the $\mathbb{Q}(q, t)$ -algebra of formal power series with basis given by $\{F_{S,n} : n \geq 0, S \subseteq [n-1]\}$. If a subset $S \subseteq [n-1]$ corresponds to a composition α , we set $F_\alpha := F_{S,n}$.

For any composition $\alpha \models n$, define a symbol \mathbf{s}_α (the *noncommutative ribbon Schur function*), formally defined to have homogeneous degree n . Let \mathbf{NSym}_n be the 2^{n-1} -dimensional $\mathbb{Q}(q, t)$ -vector space with basis $\{\mathbf{s}_\alpha : \alpha \models n\}$ and let \mathbf{NSym} be the graded vector space $\mathbf{NSym} := \bigoplus_{n \geq 0} \mathbf{NSym}_n$. The space \mathbf{NSym} is the space of *noncommutative symmetric functions*. Although there is more structure on \mathbf{NSym} (and on QSym) than the graded vector space structure (namely, they are dual graded Hopf algebras), only the vector space structure will be relevant in this paper.

Let \mathbb{F} be an arbitrary field. The *0-Hecke algebra* $H_n(0)$ of rank n over \mathbb{F} is the unital associative \mathbb{F} -algebra with generators T_1, \dots, T_{n-1} and relations

$$(2.8) \quad \begin{cases} T_i^2 = T_i & 1 \leq i \leq n-1 \\ T_i T_j = T_j T_i & |i-j| > 1 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & 1 \leq i \leq n-2. \end{cases}$$

For all $1 \leq i \leq n-1$, let $s_i := (i, i+1) \in S_n$ be the corresponding adjacent transposition. Given a permutation $\pi \in S_n$, we define $T_\pi := T_{i_1} \cdots T_{i_k} \in H_n(0)$, where $\pi = s_{i_1} \cdots s_{i_k}$ is a reduced (i.e., short as possible) expression for π as a product of adjacent transpositions. It can be shown that $\{T_\pi : \pi \in S_n\}$ is a \mathbb{F} -basis for $H_n(0)$, so that $H_n(0)$ has dimension $n!$ as a \mathbb{F} -vector space and may be viewed as a deformation of the group algebra $\mathbb{F}[S_n]$. The algebra $H_n(0)$ is not semisimple, even when the field \mathbb{F} has characteristic zero, so its representation theory has a different flavor from that of S_n .

The indecomposable projective representations of $H_n(0)$ are naturally labeled by compositions $\alpha \models n$ (see [12, 13]). For $\alpha \models n$, we let P_α denote the corresponding indecomposable projective and let

$$(2.9) \quad C_\alpha := \text{top}(P_\alpha) = P_\alpha / \text{rad}(P_\alpha)$$

be the corresponding irreducible $H_n(0)$ -module.

The Grothendieck group $G_0(H_n(0))$ is the \mathbb{Z} -module generated by all isomorphism classes $[V]$ of finite-dimensional $H_n(0)$ -modules with a relation $[V] - [U] - [W] = 0$ for every short exact sequence $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ of $H_n(0)$ -modules. The \mathbb{Z} -module $G_0(H_n(0))$ is free with basis given by (isomorphism classes of) the irreducibles $\{C_\alpha : \alpha \models n\}$. The *quasisymmetric characteristic map* Ch is defined on $G_0(H_n(0))$ by

$$(2.10) \quad \text{Ch} : C_\alpha \mapsto F_\alpha.$$

If a $H_n(0)$ -module V has composition factors $C_{\alpha(1)}, \dots, C_{\alpha(k)}$, then $\text{Ch}(V) = F_{\alpha(1)} + \dots + F_{\alpha(k)}$. Since $H_n(0)$ is not semisimple, the characteristic $\text{Ch}(V)$ does *not* determine V up to isomorphism.

Let $K_0(H_n(0))$ be the \mathbb{Z} -module generated by all isomorphism classes $[P]$ of finite-dimensional *projective* $H_n(0)$ -modules with a relation $[P] - [Q] - [R] = 0$ for every short exact sequence $0 \rightarrow Q \rightarrow P \rightarrow R \rightarrow 0$ of projective modules. The \mathbb{Z} -module $K_0(H_n(0))$ is free with basis given by (isomorphism classes of) the projective indecomposable $\{P_\alpha : \alpha \models n\}$. The *noncommutative characteristic map* \mathbf{ch} is defined on $K_0(H_n(0))$ by

$$(2.11) \quad \mathbf{ch} : P_\alpha \mapsto \mathbf{s}_\alpha,$$

This extends to give a noncommutative symmetric function $\mathbf{ch}(P)$ for any projective $H_n(0)$ -module P . The module P is determined by $\mathbf{ch}(P)$ up to isomorphism.

There are graded refinements of the maps Ch and \mathbf{ch} . Let $V = \bigoplus_{d \geq 0} V_d$ be a graded $H_n(0)$ -module with each V_d finite-dimensional. The *degree-graded quasisymmetric characteristic* is $\text{Ch}_q(V) := \sum_{d \geq 0} \text{Ch}(V_d) \cdot q^d$. If each V_d is projective, the *degree-graded noncommutative characteristic* is $\mathbf{ch}_q(V) := \sum_{d \geq 0} \mathbf{ch}(V_d) \cdot q^d$.

The quasisymmetric characteristic Ch admits a bigraded refinement as follows. The 0-Hecke algebra $H_n(0)$ has a *length filtration*

$$(2.12) \quad H_n(0)^{(0)} \subseteq H_n(0)^{(1)} \subseteq H_n(0)^{(2)} \subseteq \dots$$

where $H_n(0)^{(\ell)}$ is the subspace of $H_n(0)$ with \mathbb{F} -basis $\{T_\pi : \pi \in S_n, \text{inv}(\pi) \leq \ell\}$. If $V = H_n(0)v$ is a cyclic $H_n(0)$ -module with distinguished generator v , we get an induced length filtration of V by

$$(2.13) \quad V^{(\ell)} := H_n(0)^{(\ell)}v.$$

The *length-graded quasisymmetric characteristic* is given by

$$(2.14) \quad \text{Ch}_t(V) := \sum_{\ell \geq 0} \text{Ch}(V^{(\ell)}/V^{(\ell-1)}) \cdot t^\ell.$$

Now suppose $V = \bigoplus_{d \geq 0} V_d$ is a graded $H_n(0)$ -module which is also cyclic. We get a bifiltration of V consisting of the modules $V^{(\ell, d)} := V^{(\ell)} \cap V_d$ for $\ell, d \geq 0$. The *length-degree-bigraded quasisymmetric characteristic* is

$$(2.15) \quad \text{Ch}_{q,t}(V) := \sum_{\ell, d \geq 0} \text{Ch}(V^{(\ell, d)} / (V^{(\ell-1, d)} + V^{(\ell, d+1)})) \cdot q^d t^\ell.$$

More generally, if V is a direct sum of graded cyclic $H_n(0)$ -modules, we define $\text{Ch}_{q,t}(V)$ by applying $\text{Ch}_{q,t}$ to its cyclic summands. This may depend on the cyclic decomposition of the module V .¹

¹Our conventions for q and t in the definitions of \mathbf{ch}_q and $\text{Ch}_{q,t}$ are reversed with respect to those in [12, 13] and elsewhere. We make these conventions so as to be consistent with the case of the graded Frobenius map on S_n -modules.

2.4. Gröbner theory. A total order $<$ on the monomials in the polynomial ring $\mathbb{Q}[\mathbf{x}_n]$ is called a *term order* if

- we have $1 \leq m$ for all monomials m , and
- $m \leq m'$ implies $m \cdot m'' \leq m' \cdot m''$ for all monomials m, m', m'' .

The term order used in this paper is the *lexicographic* term order given by $x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n}$ if there exists an index i with $a_1 = b_1, \dots, a_{i-1} = b_{i-1}$ and $a_i < b_i$.

If $<$ is any term order any $f \in \mathbb{Q}[\mathbf{x}_n]$ is any nonzero polynomial, let $\text{in}_<(f)$ be the leading (i.e., greatest) term of f with respect to the order $<$. If $I \subseteq \mathbb{Q}[\mathbf{x}_n]$ is any ideal, the associated *initial ideal* is

$$(2.16) \quad \text{in}_<(I) := \langle \text{in}_<(f) : f \in I - \{0\} \rangle.$$

A finite collection $G = \{g_1, \dots, g_r\}$ of nonzero polynomials in an ideal $I \subseteq \mathbb{Q}[\mathbf{x}_n]$ is called a *Gröbner basis* of I if we have the equality of monomial ideals

$$(2.17) \quad \text{in}_<(I) = \langle \text{in}_<(g_1), \dots, \text{in}_<(g_r) \rangle.$$

If G is a Gröbner basis of I it follows that $I = \langle G \rangle$. A Gröbner basis $G = \{g_1, \dots, g_r\}$ is called *minimal* if the $<$ -leading coefficient of each g_i is 1 and $\text{in}_<(g_i) \nmid \text{in}_<(g_j)$ for all $i \neq j$. A minimal Gröbner basis $G = \{g_1, \dots, g_r\}$ is called *reduced* if in addition, for all $i \neq j$, no term of g_j is divisible by $\text{in}_<(g_i)$. After fixing a term order, every ideal $I \subseteq \mathbb{Q}[\mathbf{x}_n]$ has a unique reduced Gröbner basis.

Let $I \subseteq \mathbb{Q}[\mathbf{x}_n]$ be an ideal and let G be a Gröbner basis for I . The set of monomials in $\mathbb{Q}[\mathbf{x}_n]$

$$(2.18) \quad \{m : \text{in}_<(f) \nmid m \text{ for all } f \in I - \{0\}\} = \{m : \text{in}_<(g) \nmid m \text{ for all } g \in G\}$$

descends to a vector space basis for the quotient $\mathbb{Q}[\mathbf{x}_n]/I$. This is called the *standard monomial basis*; it is completely determined by the ideal I and the term order $<$. If I is a homogeneous ideal, the Hilbert series of $\mathbb{Q}[\mathbf{x}_n]/I$ is given by

$$(2.19) \quad \text{Hilb}(\mathbb{Q}[\mathbf{x}_n]/I; q) = \sum_m q^{\deg(m)},$$

where the sum is over all monomials in the standard monomial basis.

3. HILBERT SERIES

In this section we will derive the Hilbert series and ungraded isomorphism type of the module $R_{n,k,r}$. The method that we use dates back to Garsia and Procesi in the context of Tanisaki ideals and quotients [6].

Let $Y \subseteq \mathbb{Q}^n$ be any finite set of points and consider the ideal $\mathbf{I}(Y) \subseteq \mathbb{Q}[\mathbf{x}_n]$ of polynomials which vanish on Y . That is, we have

$$(3.1) \quad \mathbf{I}(Y) = \{f \in \mathbb{Q}[\mathbf{x}_n] : f(\mathbf{y}) = 0 \text{ for all } \mathbf{y} \in Y\}.$$

We may identify the quotient $\mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y)$ with the collection of all (polynomial) functions $Y \rightarrow \mathbb{Q}$; since Y is finite we have

$$(3.2) \quad |Y| = \dim(\mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y)).$$

If Y is stable under the coordinate permutation action of S_n , we have the further identification of S_n -modules

$$(3.3) \quad \mathbb{Q}[Y] \cong_{S_n} \mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y).$$

The ideal $\mathbf{I}(Y)$ is usually not homogeneous; we wish to replace it by a homogeneous ideal so that the associated quotient is graded. For any nonzero polynomial $f \in \mathbf{I}(X)$, write $f = f_d + \cdots + f_1 + f_0$

where f_i is homogeneous of degree i and $f_d \neq 0$. Let $\tau(f) = f_d$ be the top homogeneous component of f . The ideal $\mathbf{T}(Y) \subseteq \mathbb{Q}[\mathbf{x}_n]$ is given by

$$(3.4) \quad \mathbf{T}(Y) := \langle \tau(f) : f \in \mathbf{I}(Y) - \{0\} \rangle.$$

By construction the ideal $\mathbf{T}(Y)$ is homogeneous, so that the quotient $\mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y)$ is graded. Furthermore, we still have the dimension equality

$$(3.5) \quad |Y| = \dim(\mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y)) = \dim(\mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y))$$

and the S_n -module isomorphism

$$(3.6) \quad \mathbb{Q}[Y] \cong_{S_n} \mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y) \cong_{S_n} \mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y)$$

whenever the point set Y is S_n -stable.

The symmetric group S_n acts on $S_{n,k,r}$ by permuting the positive letters $1, 2, \dots, n$. We aim to prove that $R_{n,k,r} \cong \mathbb{Q}[S_{n,k,r}]$ as ungraded S_n -modules. To do this, our strategy is as follows.

- (1) Find a point set $Y_{n,k,r} \subseteq \mathbb{Q}^n$ which is stable under the action of S_n such that there is a S_n -equivariant bijection from $Y_{n,k,r}$ to $S_{n,k,r}$.
- (2) Prove that $I_{n,k,r} \subseteq \mathbf{T}(Y_{n,k,r})$ by showing that the generators of $I_{n,k,r}$ arise as top degree components of polynomials in $\mathbf{I}(Y_{n,k,r})$.
- (3) Prove that

$$\dim(R_{n,k,r}) = \dim(\mathbb{Q}[\mathbf{x}_n]/I_{n,k,r}) \leq |S_{n,k,r}| = \dim(\mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k,r}))$$

and use the relation $I_{n,k,r} \subseteq \mathbf{T}(Y_{n,k,r})$ to conclude that $I_{n,k,r} = \mathbf{T}(Y_{n,k,r})$.

The point set $Y_{n,k,r}$ which accomplishes Step 1 is the following.

Definition 3.1. Fix $n+k$ distinct rational numbers $\alpha_1, \alpha_2, \dots, \alpha_{n+k} \in \mathbb{Q}$. Let $Y_{n,k,r}$ be the set of points $(y_1, y_2, \dots, y_n) \in \mathbb{Q}^n$ such that

- the coordinates y_1, y_2, \dots, y_n are distinct and lie in $\{\alpha_1, \alpha_2, \dots, \alpha_{n+k}\}$, and
- the numbers $\alpha_{n+k-r+1}, \dots, \alpha_{n+k-1}, \alpha_{n+k}$ all appear as coordinates (y_1, y_2, \dots, y_n) .

It is clear that $Y_{n,k,r}$ is stable under the action of S_n . We have a natural identification of $Y_{n,k,r}$ with permutations in $S_{n,k,r}$ given by letting a copy of α_i in position j of (y_1, \dots, y_n) correspond to the letter j in position i of the corresponding permutation in $S_{n,k,r}$. For example, if $(n, k, r) = (4, 3, 2)$ then

$$(\alpha_7, \alpha_2, \alpha_4, \alpha_6) \leftrightarrow 0203041.$$

This bijection $Y_{n,k,r} \leftrightarrow S_{n,k,r}$ is clearly S_n -equivariant, so Step 1 of our strategy is accomplished. Step 2 of our strategy is achieved in the following lemma.

Lemma 3.2. We have $I_{n,k,r} \subseteq \mathbf{T}(Y_{n,k,r})$.

Proof. We show that every generator of $I_{n,k,r}$ arises as the leading term of a polynomial in $\mathbf{I}(Y_{n,k,r})$. We begin with the elementary symmetric function generators $e_{n-r+1}(\mathbf{x}_n), \dots, e_{n-1}(\mathbf{x}_n), e_n(\mathbf{x}_n)$. Consider the rational function in t given by

$$(3.7) \quad \frac{(1-x_1t)(1-x_2t) \cdots (1-x_nt)}{(1-\alpha_{n+k-r+1}t) \cdots (1-\alpha_{n+k-1}t)(1-\alpha_{n+k}t)} = \sum_{i,j \geq 0} (-1)^i e_i(\mathbf{x}_n) h_j(\alpha_{n+k-r+1}, \dots, \alpha_{n+k}) \cdot t^{i+j}.$$

If $(x_1, \dots, x_n) \in Y_{n,k,r}$, the r factors in the denominator cancel with r factors in the numerator, so that this rational expression is a polynomial in t of degree $n-r$. In particular, for $n-r+1 \leq m \leq r$ taking the coefficient of t^m on both sides gives

$$(3.8) \quad \sum_{i=0}^m (-1)^i e_{m-i}(\mathbf{x}_n) h_i(\alpha_{n+k-r+1}, \dots, \alpha_{n+k}) \in \mathbf{I}(Y_{n,k,r}),$$

so that $e_m(\mathbf{x}_n) \in \mathbf{T}(Y_{n,k,r})$.

A similar trick shows that the homogeneous symmetric functions $h_{k+1}(\mathbf{x}_n), \dots, h_{k+n}(\mathbf{x}_n)$ lie in $\mathbf{T}(Y_{n,k,r})$. Consider the rational function

$$(3.9) \quad \frac{(1 - \alpha_1 t)(1 - \alpha_2 t) \cdots (1 - \alpha_{n+k} t)}{(1 - x_1 t)(1 - x_2 t) \cdots (1 - x_n t)} = \sum_{i,j \geq 0} (-1)^j h_i(\mathbf{x}_n) e_j(\alpha_1, \dots, \alpha_{n+k}) t^{i+j}.$$

If $(x_1, \dots, x_n) \in Y_{n,k,r}$ the n factors in the denominator cancel with n factors in the numerator, giving a polynomial in t of degree k . For $m \geq k+1$, taking the coefficient of t^m on both sides gives

$$(3.10) \quad \sum_{i=0}^m (-1)^i h_{m-i}(\mathbf{x}_n) e_i(\alpha_1, \dots, \alpha_{n+k}) \in \mathbf{I}(Y_{n,k,r}),$$

so that $h_m(\mathbf{x}_n) \in \mathbf{T}(Y_{n,k,r})$. \square

Step 3 of our strategy will take more work. To begin, we identify a convenient collection of monomials in the initial ideal $\text{in}_{<}(I_{n,k,r})$ with respect to the lexicographic term order. Given a subset $S = \{s_1 < \cdots < s_m\} \subseteq [n]$ the corresponding *skip monomial* $\mathbf{x}(S)$ is given by

$$(3.11) \quad \mathbf{x}(S) := x_{s_1}^{s_1} x_{s_2}^{s_2-1} \cdots x_{s_m}^{s_m-m+1}.$$

In particular, if $n = 8$ we have $\mathbf{x}(2458) = x_2^2 x_4^3 x_5^3 x_8^5$.

Lemma 3.3. *Let $<$ be the lexicographic term order on $\mathbb{Q}[\mathbf{x}_n]$. If $S \subseteq [n]$ satisfies $|S| = n - r + 1$ we have $\mathbf{x}(S) \in \text{in}_{<}(I_{n,k,r})$. Moreover, we have $x_1^{k+1}, x_2^{k+2}, \dots, x_n^{k+n} \in \text{in}_{<}(I_{n,k,r})$.*

Proof. The assertion regarding skip monomials comes from combining [10, Lem. 3.4] (and in particular [10, Eqn. 3.5]) and [10, Lem. 3.5]. To prove the second assertion, the identities

$$(3.12) \quad h_{m+1}(x_i, x_{i+1}, \dots, x_n) - x_i \cdot h_m(x_i, x_{i+1}, \dots, x_n) = h_{m+1}(x_{i+1}, \dots, x_n)$$

(for $1 \leq i \leq n$ and $m \geq 0$) imply that

$$(3.13) \quad h_{k+1}(x_1, \dots, x_n), h_{k+2}(x_2, \dots, x_n), \dots, h_{k+n}(x_n) \in I_{n,k,r},$$

so that $x_1^{k+1}, x_2^{k+2}, \dots, x_n^{k+n} \in \text{in}_{<}(I_{n,k,r})$. \square

The initial terms provided by Lemma 3.3 will be all we need. We name the monomials $m \in \mathbb{Q}[\mathbf{x}_n]$ which are not divisible by any of these initial terms as follows.

Definition 3.4. *A monomial $m \in \mathbb{Q}[\mathbf{x}_n]$ is (n, k, r) -good if*

- *we have $\mathbf{x}(S) \nmid m$ for all $S \subseteq [n]$ with $|S| = n - r + 1$, and*
- *we have $x_i^{k+i} \nmid m$ for all $1 \leq i \leq n$.*

Let $\mathcal{M}_{n,k,r}$ denote the set of all (n, k, r) -good monomials.

By Lemma 3.3, the monomials in $\mathcal{M}_{n,k,r}$ contain the standard monomial basis of $R_{n,k,r}$, and so descend to a spanning set of $R_{n,k,r}$. We will see that $\mathcal{M}_{n,k,r}$ is in fact that standard monomial basis of $R_{n,k,r}$. We will do this using the following combinatorial result.

Lemma 3.5. *There is an injection $\Psi : S_{n,k,r} \rightarrow \mathcal{M}_{n,k,r}$ with the property that $\deg(\Psi(\pi)) = \text{inv}(\pi)$ for all $\pi \in S_{n,k,r}$.*

It will develop that the map Ψ of Lemma 3.5 is actually a bijection.

Proof. The map Ψ will essentially be the inversion code. Let $\pi = \pi_1 \dots \pi_{n+k} \in S_{n,k,r}$ be a r -tail positive permutation of the multiset $\{0^k, 1, 2, \dots, n\}$. The *code* of π is the sequence (c_1, \dots, c_n) where

$$(3.14) \quad c_i = \text{the number of letters } 0, 1, 2, \dots, i-1 \text{ to the right of } i \text{ in } \pi.$$

For example, if $\pi = 40130052$ the code is $(c_1, c_2, c_3, c_4, c_5) = (2, 0, 3, 6, 1)$. It is clear that the sum of the code of π gives the inversion number $\text{inv}(\pi)$. If $\pi \in S_{n,k,r}$ has code (c_1, \dots, c_n) , we define $\Psi(\pi) := x_1^{c_1} \cdots x_n^{c_n}$.

We argue that Ψ is a well defined function $S_{n,k,r} \rightarrow \mathcal{M}_{n,k,r}$, that is, we have $\Psi(\pi) \in \mathcal{M}_{n,k,r}$ for all $\pi \in S_{n,k,r}$. Let $\pi \in S_{n,k,r}$ have code (c_1, \dots, c_n) . Since π contains k copies of 0, it is clear that $c_i < k + i$ for all $1 \leq i \leq n$, so that $x_i^{k+i} \nmid \Psi(\pi)$ for all $1 \leq i \leq n$.

Now let $S = \{s_1 < \cdots < s_{n-r+1}\} \subseteq [n]$ and suppose $\mathbf{x}(S) \mid \Psi(\pi)$. This means that $c_{s_i} \geq s_i - i + 1$ for all $1 \leq i \leq n - r + 1$. Let $T = \{\pi_{n+k-r+1}, \dots, \pi_{n+k-1}, \pi_{n+k}\}$ be the r -tail of π ; since $\pi \in S_{n,k,r}$ the set T consists of r positive numbers. We argue that $S \cap T = \emptyset$ as follows.

- If $s_1 \in T$ we would have $c_{s_1} \leq s_1 - 1$ (since s_1 could form inversions with only $1, 2, \dots, s_1 - 1$), contradicting the inequality $c_{s_i} \geq s_i$. We conclude that $s_1 \notin T$.
- If $s_1, \dots, s_{i-1} \notin T$ and $s_i \in T$, we would have $c_{s_i} \leq s_i - i$ (since s_i can only form inversions with those letters in $1, 2, \dots, s_i - 1$ which lie in T), contradicting the inequality $c_{s_i} \geq s_i - i + 1$. We conclude that $s_i \notin T$.

Induction gives the result that $S \cap T = \emptyset$. However, this contradicts the facts that $|S| = n - r + 1$, $|T| = r$, and that there are a total of n positive letters in π . This concludes the proof that the map $\Psi : S_{n,k,r} \rightarrow \mathcal{M}_{n,k,r}$ is well defined.

The relation $\deg(\Psi(\pi)) = \text{inv}(\pi)$ is clear from construction. The fact that Ψ is an injection is equivalent to the fact that a permutation $\pi = \pi_1 \dots \pi_{n+k} \in S_{n,k,r}$ is determined by its code (c_1, \dots, c_n) . This assertion is true more broadly for any permutation of the multiset $\{0^k, 1, 2, \dots, n\}$ (whether or not it is r -tail positive); we leave the verification to the reader. \square

We are ready to derive the Hilbert series of $R_{n,k,r}$.

Theorem 3.6. *Endow monomials in $\mathbb{Q}[\mathbf{x}_n]$ with the lexicographic term order. The standard monomial basis of $R_{n,k,r}$ is $\mathcal{M}_{n,k,r}$. The Hilbert series of $R_{n,k,r}$ is given by*

$$(3.15) \quad \text{Hilb}(R_{n,k,r}; q) = \begin{bmatrix} n + k - r \\ k \end{bmatrix}_q \cdot [n]!_q.$$

Proof. Let \mathcal{B}^T be the standard monomial basis of $\mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k,r})$ and let \mathcal{B}^J be the standard monomial basis of $R_{n,k,r} = \mathbb{Q}[\mathbf{x}_n]/I_{n,k,r}$. We know that $|S_{n,k,r}| = |\mathcal{B}^T|$. Lemma 3.2 implies that $\mathcal{B}^T \subseteq \mathcal{B}^J$. Lemma 3.3 further implies the containment $\mathcal{B}^J \subseteq \mathcal{M}_{n,k,r}$. Finally, Lemma 3.5 gives the relation $|\mathcal{M}_{n,k,r}| \leq |S_{n,k,r}|$. Putting these facts together gives

$$(3.16) \quad \mathcal{B}^T = \mathcal{B}^J = \mathcal{M}_{n,k,r},$$

and the fact that all of these sets have size $|S_{n,k,r}|$. In particular, the standard monomial basis of $R_{n,k,r}$ is $\mathcal{M}_{n,k,r}$.

By the last paragraph, the map Ψ of Lemma 3.5 is a bijection. It follows that

$$(3.17) \quad \text{Hilb}(R_{n,k,r}; q) = \sum_{m \in \mathcal{M}_{n,k,r}} q^{\deg(m)} = \sum_{\pi \in S_{n,k,r}} q^{\text{inv}(\pi)}.$$

It is well known that $\sum_{\pi \in S_n} q^{\text{inv}(\pi)} = [n]!_q$. The q -binomial coefficient in

$$(3.18) \quad \sum_{\pi \in S_{n,k,r}} q^{\text{inv}(\pi)} = \begin{bmatrix} n + k - r \\ k \end{bmatrix}_q \cdot [n]!_q$$

comes from the ways of inserting k copies of 0 among the first $n - r$ letters of a permutation in S_n . \square

We can also derive the ungraded S_n -isomorphism type of the quotient $R_{n,k,r}$.

Corollary 3.7. *As an ungraded S_n -module we have $R_{n,k,r} \cong_{S_n} \mathbb{Q}[S_{n,k,r}]$.*

Proof. Lemma 3.2 and Theorem 3.6 give the isomorphisms

$$(3.19) \quad R_{n,k,r} \cong \mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k,r}) \cong \mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y_{n,k,r}) \cong \mathbb{Q}[S_{n,k,r}]$$

of ungraded S_n -modules. \square

We describe a minimal Gröbner basis for the ideal $I_{n,k,r}$. Given a subset $S = \{s_1 < \dots < s_m\} \subseteq [n]$, let $\gamma(S) = (\gamma(S)_1, \dots, \gamma(S)_n)$ be the length n *skip vector* of nonnegative integers given by

$$(3.20) \quad \gamma(S)_i = \begin{cases} s_j - j + 1 & i = s_j \\ 0 & i \notin S. \end{cases}$$

Let $\gamma(S)^* = (\gamma(S)_n, \dots, \gamma(S)_1)$ be the reversal of the vector $\gamma(S)$. If $\gamma = (\gamma_1, \dots, \gamma_n)$ is any length n vector of nonnegative integers, let $\kappa_\gamma(\mathbf{x}_n) \in \mathbb{Q}[\mathbf{x}_n]$ be the associated *Demazure character* (see [10, Sec. 2] for its definition). Finally, if $f(\mathbf{x}_n) \in \mathbb{Q}[\mathbf{x}_n]$ is any polynomial, let $f(\mathbf{x}_n^*)$ be the polynomial obtained by reversing the variables in $f(\mathbf{x}_n)$ so that

$$(3.21) \quad f(\mathbf{x}_n^*) = f(x_n, x_{n-1}, \dots, x_1).$$

Corollary 3.8. *Endow monomials in $\mathbb{Q}[\mathbf{x}_n]$ with the lexicographic term order. A Gröbner basis for the ideal $I_{n,k,r}$ consists of the polynomials*

$$h_{k+1}(x_1, x_2, \dots, x_n), h_{k+2}(x_2, \dots, x_n), \dots, h_{k+n}(x_n)$$

together with the polynomials

$$\kappa_{\gamma(S)^*}(\mathbf{x}_n^*),$$

where S ranges over all $n - r + 1$ -element subsets of $[n]$. When $r < n$ and $k > 0$ this Gröbner basis is minimal.

The Gröbner basis in Corollary 3.8 is typically not reduced.

Proof. The proof of Lemma 3.3 shows that the polynomial $h_{k+i}(x_i, x_{i+1}, \dots, x_n)$ lies in the ideal $I_{n,k,r}$. By [10, Lem. 3.4] (and in particular [10, Eqn. 3.5]) shows that the relevant variable reversed Demazure characters lie in $I_{n,k,r}$.

Let $<$ be the lexicographic term order on $\mathbb{Q}[\mathbf{x}_n]$. We have $\text{in}_{<}(h_{k+i}(x_i, x_{i+1}, \dots, x_n)) = x_i^{k+i}$ and $\text{in}_{<}(\kappa_{\gamma(S)^*}(\mathbf{x}_n^*) = \mathbf{x}(S)$ (see [10, Lem. 3.5]). We know that these initial terms generate $\text{in}_{<}(I_{n,k,r})$, proving the assertion about the claimed collection of polynomials being a Gröbner basis. When $r < n$ and $k > 0$, none of the relevant skip monomials $\mathbf{x}(S)$ are divisible by any of the variable powers $x_1^{k+1}, \dots, x_n^{k+n}$. This proves the claim about minimality. \square

For example, consider the case $(n, k, r) = (5, 2, 3)$. A minimal Gröbner basis for $J_{5,2,3}$ is given by the polynomials

$$h_3(x_1, x_2, x_3, x_4, x_5), h_4(x_2, x_3, x_4, x_5), h_5(x_3, x_4, x_5), h_6(x_4, x_5), h_7(x_5)$$

together with the variable reversed Demazure characters

$$\begin{aligned} &\kappa_{(0,0,1,1,1)}(\mathbf{x}_5^*), \quad \kappa_{(0,2,0,1,1)}(\mathbf{x}_5^*), \quad \kappa_{(3,0,0,1,1)}(\mathbf{x}_5^*), \quad \kappa_{(0,2,2,0,1)}(\mathbf{x}_5^*), \quad \kappa_{(3,0,2,0,1)}(\mathbf{x}_5^*), \\ &\kappa_{(3,3,0,0,1)}(\mathbf{x}_5^*), \quad \kappa_{(0,2,2,2,0)}(\mathbf{x}_5^*), \quad \kappa_{(3,0,2,2,0)}(\mathbf{x}_5^*), \quad \kappa_{(3,3,0,2,0)}(\mathbf{x}_5^*), \quad \kappa_{(3,3,3,0,0)}(\mathbf{x}_5^*). \end{aligned}$$

Theorem 3.6 describes the standard monomial basis $\mathcal{M}_{n,k,r}$ of $R_{n,k,r}$ in terms of divisibility by skip monomials. However, a more direct characterization of this standard monomial basis is available. Let $k \geq 0$ and $r \leq n$. For any $(n - r)$ -element subset $T \subseteq [n]$, define a length n sequence $\delta(T) := (\delta(T)_1, \dots, \delta(T)_n)$ by the formula

$$(3.22) \quad \delta(T)_i := \begin{cases} i + k - 1 & i \in T \\ j - 1 & i \notin T \text{ and } i = s_j, \end{cases}$$

where $[n] - T = \{s_1 < \dots < s_r\}$. Any of the $\binom{n}{r}$ sequences which can be obtained in this way is an (n, k, r) -staircase. For example, the $(5, 2, 3)$ -staircases are

$$(0, 1, 2, 5, 6), \quad (0, 1, 4, 2, 6), \quad (0, 3, 1, 2, 6), \quad (2, 0, 1, 2, 6), \quad (0, 1, 4, 5, 2), \\ (0, 3, 1, 5, 2), \quad (2, 0, 1, 5, 2), \quad (0, 3, 4, 1, 2), \quad (2, 0, 4, 1, 2), \quad (2, 3, 0, 1, 2).$$

Proposition 3.9. *Endow monomials in $\mathbb{Q}[\mathbf{x}_n]$ with the lexicographic term order. The standard monomial basis $\mathcal{M}_{n,k,r}$ of $R_{n,k,r}$ consists of those monomials in $\mathbb{Q}[\mathbf{x}_n]$ whose exponent vectors are componentwise \leq at least one (n, k, r) -staircase.*

Proof. Let $\mathcal{N}_{n,k,r}$ be the collection of monomials in $\mathbb{Q}[\mathbf{x}_n]$ whose exponent vectors are componentwise \leq at least one (n, k, r) -staircase. If $\delta_n(T) = (a_1, \dots, a_n)$ is an (n, k, r) -staircase for some $(n - r)$ -element set $T \subseteq [n]$ and $m = x_1^{a_1} \dots x_n^{a_n}$ is the corresponding monomial, it is clear that $a_i < k + i$ for all i , so that $x_i^{k+i} \nmid m$. If $S \subseteq [n]$ satisfies $|S| = n - r + 1$ then at least one index $i \in S$ satisfies $i \notin T$, which forces $\mathbf{x}(S) \nmid m$. It follows that $\mathcal{N}_{n,k,r} \subseteq \mathcal{M}_{n,k,r}$.

On the other hand, we may construct a map

$$(3.23) \quad \Phi : S_{n,k,r} \rightarrow \mathcal{N}_{n,k,r}$$

by letting $\Phi(\pi) = (c_1, \dots, c_n)$ be the code of any r -tail positive permutation $\pi \in S_{n,k,r}$. The fact that π is r -tail positive implies that $\Phi(\pi) \in \mathcal{N}_{n,k,r}$, so that Φ is well defined. It is clear that Φ is injective, so that

$$(3.24) \quad |S_{n,k,r}| \leq |\mathcal{N}_{n,k,r}| \leq |\mathcal{M}_{n,k,r}| = |S_{n,k,r}|$$

and we have $\mathcal{N}_{n,k,r} = \mathcal{M}_{n,k,r}$, as desired. \square

For example, if $(n, k, r) = (2, 2, 1)$ the $(2, 2, 1)$ -staircases are $(0, 3)$ and $(2, 0)$ so that

$$\mathcal{M}_{2,2,1} = \{1, x_1, x_1^2, x_2, x_2^2, x_2^3\}.$$

4. FROBENIUS SERIES

In this section we derive the Frobenius series of $R_{n,k,r}$. Our first lemma is a short exact sequence which establishes a Pascal-type recursion for $\text{grFrob}(R_{n,k,r}; q)$.

Lemma 4.1. *Suppose $n, k, r \geq 0$ with $r < n$ and $k > 0$. There is a short exact sequence of S_n -modules*

$$(4.1) \quad 0 \rightarrow R_{n,k-1,r} \rightarrow R_{n,k,r} \rightarrow R_{n,k,r+1} \rightarrow 0,$$

where the first map is homogeneous of degree $n - r$ and the second map is homogeneous of degree 0. Equivalently, we have the equality of graded Frobenius characters

$$(4.2) \quad \text{grFrob}(R_{n,k,r}; q) = \text{grFrob}(R_{n,k,r+1}; q) + q^{n-r} \cdot \text{grFrob}(R_{n,k-1,r}; q).$$

Proof. We have the inclusion of ideals $I_{n,k,r} \subseteq I_{n,k,r+1}$; we let the second map be the canonical projection $\pi : R_{n,k,r} \twoheadrightarrow R_{n,k,r+1}$. We have a homogeneous map $\tilde{\varphi} : \mathbb{Q}[\mathbf{x}_n] \rightarrow R_{n,k,r}$ of degree $n - r$ given by multiplication by $e_{n-r}(\mathbf{x}_n)$, and then projecting onto $R_{n,k,r}$.

We claim that $\tilde{\varphi}(I_{n,k-1,r}) = 0$, so that $\tilde{\varphi}$ induces a well defined map $\varphi : R_{n,k-1,r} \rightarrow R_{n,k,r}$. This is equivalent to showing that $h_k(\mathbf{x}_n) \cdot e_{n-r}(\mathbf{x}_n) \in I_{n,k,r}$. The Pieri Rule implies that

$$(4.3) \quad h_k(\mathbf{x}_n) \cdot e_{n-r}(\mathbf{x}_n) = s_{(k, 1^{n-r})}(\mathbf{x}_n) + s_{(k+1, 1^{n-r-1})}(\mathbf{x}_n);$$

we will show that both terms on the right hand side lie in $I_{n,k,r}$.

To see that $s_{(k, 1^{n-r})}(\mathbf{x}_n) \in I_{n,k,r}$, observe that, for $1 \leq i \leq r$ we have

$$(4.4) \quad h_{k-r+i}(\mathbf{x}_n) \cdot e_{n-i+1}(\mathbf{x}_n) = s_{(k-r+i, 1^{n-i+1})}(\mathbf{x}_n) + s_{(k-r+i+1, 1^{n-i})}(\mathbf{x}_n) \in I_{n,k,r}.$$

It follows that modulo $I_{n,k,r}$ we have the congruences

$$(4.5) \quad s_{(k, 1^{n-r})}(\mathbf{x}_n) \equiv -s_{(k+1, 1^{n-r-1})}(\mathbf{x}_n) \equiv s_{(k+2, 1^{n-r-2})}(\mathbf{x}_n) \equiv \dots \equiv \pm s_{(k+n-r)}(\mathbf{x}_n) \equiv 0,$$

where the last congruence used the fact that $s_{(k+n-r)}(\mathbf{x}_n) = h_{k+n-r}(\mathbf{x}_n) \in I_{n,k,r}$ since $r < n$. This chain of congruences also shows that $s_{(k+1,1^{n-r-1})}(\mathbf{x}_n) \in I_{n,k,r}$.

By the last paragraph, we have a well defined induced map $\varphi : R_{n,k-1,r} \rightarrow R_{n,k,r}$. It is clear that $\text{Im}(\varphi) = \text{Ker}(\pi)$. Moreover, the Pascal relation implies that

$$(4.6) \quad |S_{n,k-1,r}| + |S_{n,k,r+1}| = |S_{n,k,r}|,$$

so that by Theorem 3.6 we have

$$(4.7) \quad \dim(R_{n,k-1,r}) + \dim(R_{n,k,r+1}) = \dim(R_{n,k,r}).$$

Since π is a surjection, this forces the sequence

$$(4.8) \quad 0 \rightarrow R_{n,k-1,r} \xrightarrow{\varphi} R_{n,k,r} \xrightarrow{\pi} R_{n,k,r+1} \rightarrow 0$$

to be exact. To finish the proof, observe that the maps φ and π commute with the action of S_n . \square

We are ready to state the graded Frobenius image of $R_{n,k,r}$. We will give several formulas for this image. For any word w over the nonnegative integers, define the monomial \mathbf{x}^w to be

$$(4.9) \quad \mathbf{x}^w := x_1^{\# \text{ of 1's in } w} x_2^{\# \text{ of 2's in } w} \dots;$$

in particular, any copies of 0 in w do not affect \mathbf{x}^w .

Theorem 4.2. *The graded Frobenius image of $R_{n,k,r}$ is given by*

$$(4.10) \quad \text{grFrob}(R_{n,k,r}; q) = \left[\begin{matrix} n+k-r \\ k \end{matrix} \right]_q \cdot \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T)} \cdot s_{\text{shape}(T)}$$

$$(4.11) \quad = \sum_w q^{\text{inv}(w)} \mathbf{x}^w.$$

The last sum ranges over all length $n+k$ words $w = w_1 \dots w_{n+k}$ in the alphabet of nonnegative integers which contain precisely k copies of 0 and are r -tail positive.

Proof. By considering the placement of the k copies of 0 in a r -tail positive word w appearing in the final sum, we see that

$$(4.12) \quad \sum_w q^{\text{inv}(w)} \mathbf{x}^w = \left[\begin{matrix} n+k-r \\ k \end{matrix} \right]_q \cdot \sum_{\substack{v=v_1 \dots v_n \\ v_i \in \mathbb{Z}_{>0}}} q^{\text{inv}(v)} \mathbf{x}^v.$$

On the other hand, we have

$$(4.13) \quad \sum_{\substack{v=v_1 \dots v_n \\ v_i \in \mathbb{Z}_{>0}}} q^{\text{inv}(v)} \mathbf{x}^v = \sum_{\substack{v=v_1 \dots v_n \\ v_i \in \mathbb{Z}_{>0}}} q^{\text{maj}(v)} \mathbf{x}^v = \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T)} \cdot s_{\text{shape}(T)} = \text{grFrob}(R_n; q),$$

where the first equality uses the equidistribution of the statistics inv and maj on permutations of a fixed multiset of positive integer, the second follows from standard properties of the RSK correspondence, and the third is a consequence of the work of Lusztig-Stanley [16].

By the last paragraph, it suffices to prove the first equality asserted in the statement of the theorem. If $r \geq n$ or $k = 0$ then $R_{n,k,r} = R_n$ and this equality is trivial. Otherwise, we have the q -Pascal relation

$$(4.14) \quad \left[\begin{matrix} n+k-r \\ k \end{matrix} \right]_q = \left[\begin{matrix} n+k-r-1 \\ k \end{matrix} \right]_q + q^{n-r} \cdot \left[\begin{matrix} n+k-r-1 \\ k-1 \end{matrix} \right]_q,$$

so that the theorem follows from Lemma 4.1 and induction. \square

The short exact sequence in Lemma 4.1 gives a recipe for constructing bases of $R_{n,k,r}$ from bases of the classical coinvariant algebra R_n . We switch from working over \mathbb{Q} to working over an arbitrary field \mathbb{F} , so that the ideals $I_{n,k,r}, I_n$ are defined inside the ring $\mathbb{F}[\mathbf{x}_n] := \mathbb{F}[x_1, \dots, x_n]$ and we have $R_{n,k,r} := \mathbb{F}[\mathbf{x}_n]/I_{n,k,r}, R_n := \mathbb{F}[\mathbf{x}_n]/I_n$.

Theorem 4.3. *Let $\mathcal{C}_n = \{b_\pi(\mathbf{x}_n) : \pi \in S_n\}$ be a collection of polynomials in $\mathbb{F}[\mathbf{x}_n]$ indexed by permutations in S_n which descends to a basis of R_n . The collection of polynomials*

$$(4.15) \quad \mathcal{C}_{n,k,r} := \{b_\pi(\mathbf{x}_n) \cdot e_\lambda(\mathbf{x}_n) : \pi \in S_n, \lambda_1 \leq n-r, \text{ and } \lambda \text{ has } \leq k \text{ parts}\}$$

in $\mathbb{F}[\mathbf{x}_n]$ descends to a basis of $R_{n,k,r}$.

Proof. This is trivial when $k = 0$ or $r \geq n$, so we assume $k > 0$ and $r < n$.

The arguments of Section 3 apply to show that $\dim(R_{n,k,r}) = |S_{n,k,r}|$ when working over the arbitrary field \mathbb{F} .² The proof of Lemma 4.1 then applies over \mathbb{F} to give a short exact sequence of graded \mathbb{F} -vector spaces

$$(4.16) \quad 0 \rightarrow R_{n,k-1,r} \xrightarrow{\cdot e_{n-r}(\mathbf{x}_n)} R_{n,k,r} \xrightarrow{\pi} R_{n,k,r+1} \rightarrow 0,$$

where π is the canonical projection. We may inductively assume that $\mathcal{C}_{n,k-1,r}$ descends to a \mathbb{F} -basis of $R_{n,k-1,r}$ and that $\mathcal{C}_{n,k,r+1}$ descends to a \mathbb{F} -basis of $S_{n,k,r+1}$. Exactness implies that

$$(4.17) \quad \{f(\mathbf{x}_n) : f(\mathbf{x}_n) \in \mathcal{C}_{n,k,r+1}\} \cup \{g(\mathbf{x}_n) \cdot e_{n-r}(\mathbf{x}_n) : g(\mathbf{x}_n) \in \mathcal{C}_{n,k-1,r}\} = \mathcal{C}_{n,k,r}$$

descends to an \mathbb{F} -basis of $R_{n,k,r}$. \square

Theorem 4.3 reinforces the fact that $R_{n,k,r}$ consists of $\binom{n+k-r}{k}$ copies of R_n , graded by the q -binomial coefficient $\left[\begin{smallmatrix} n+k-r \\ k \end{smallmatrix} \right]_q$. Interesting bases \mathcal{C}_n to which Theorem 4.3 can be applied include

- the *Artin basis* [2]

$$(4.18) \quad \mathcal{C}_n = \{x_1^{i_1} \cdots x_n^{i_n} : 0 \leq i_j < j\}$$

(which is connected to the *inv* statistic on permutations in S_n) and

- the *Garsia-Stanton basis* (or the *descent monomial basis* [5, 7] $\mathcal{C}_n = \{gs_\pi : \pi \in S_n\}$ where

$$(4.19) \quad gs_\pi = \prod_{\pi_i > \pi_{i+1}} x_{\pi_1} \cdots x_{\pi_i}$$

(which is connected to the *maj* statistic on permutations in S_n).

The GS basis above can be deformed somewhat to describe the isomorphism type of $R_{n,k,r}$ as a module over the 0-Hecke algebra. The algebra $H_n(0)$ acts on the polynomial ring $\mathbb{F}[\mathbf{x}_n]$ by letting the generator T_i act by the *Demazure operator* σ_i , where

$$(4.20) \quad \sigma_i \cdot f := \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}.$$

Here $s_i(f)$ is the polynomial obtained by interchanging x_i and x_{i+1} in $f(\mathbf{x}_n)$. It can be shown that if $f \in \mathbb{F}[\mathbf{x}_n]^{S_n}$ is any symmetric polynomial and $g \in \mathbb{F}[\mathbf{x}_n]$ is an arbitrary polynomial then

$$(4.21) \quad \sigma_i(fg) = f\sigma_i(g).$$

Therefore, any ideal $I \subseteq \mathbb{F}[\mathbf{x}_n]$ generated by symmetric polynomials is stable under the action of $H_n(0)$. In particular, the ideal $I_{n,k,r}$ is stable under the action of $H_n(0)$, and the quotient $R_{n,k,r} = \mathbb{F}[\mathbf{x}_n]/I_{n,k,r}$ carries the structure of an $H_n(0)$ -module.

Huang [12] studied the coinvariant ring R_n as a graded module over the 0-Hecke algebra $H_n(0)$. We apply Theorem 4.3 to generalize Huang's results to the quotient $R_{n,k,r}$. If V is any graded $H_n(0)$ -module, we let $V(i)$ denote the graded $H_n(0)$ -module with components $V(i)_j := V_{i+j}$.

²If \mathbb{F} is a finite field, there might not be enough elements in \mathbb{F} for the point set $Y_{n,k,r}$ of Definition 3.1 to make sense. To get around this, we may apply [13, Lem. 3.1] to harmlessly replace \mathbb{F} by an extension field \mathbb{K} .

Theorem 4.4. *Let n, k , and r be nonnegative integers with $r \leq n$. We have an isomorphism of graded $H_n(0)$ -modules*

$$(4.22) \quad R_{n,k,r} \cong \bigoplus_{\lambda \subseteq (n-r) \times k} R_n(-|\lambda|).$$

Here the direct sum is over all partitions λ which satisfy $\lambda_1 \leq n-r$ and have at most k parts. The module $R_n = \mathbb{F}[\mathbf{x}_n]/I_n$ is the coinvariant algebra viewed as a graded $H_n(0)$ -module.

Proof. Huang [12] introduced the following modified GS basis of R_n . For $1 \leq i \leq n-1$, define an operator $\bar{\sigma}_i$ on $\mathbb{F}[\mathbf{x}_n]$ by the rule $\bar{\sigma}_i := \sigma_i - 1$. For any permutation $\pi \in S_n$, define $\bar{\sigma}_\pi := \bar{\sigma}_{i_1} \cdots \bar{\sigma}_{i_k}$ where $s_{i_1} \cdots s_{i_k}$ is any reduced word for π . Finally, given $\pi \in S_n$, let $\mathbf{x}_{\text{Des}(\pi)}$ be the monomial

$$(4.23) \quad \mathbf{x}_{\text{Des}(\pi)} := \prod_{i \in \text{Des}(\pi)} (x_1 x_2 \cdots x_i).$$

For example, we have $\mathbf{x}_{21543} = (x_1) \cdot (x_1 x_2 x_3) \cdot (x_1 x_2 x_3 x_4)$. Huang proves [12, Thm. 4.5] that the collection of polynomials

$$(4.24) \quad \mathcal{C}_n := \{\bar{\sigma}_\pi \cdot \mathbf{x}_{\text{Des}(\pi)} : \pi \in S_n\}$$

in $\mathbb{F}[\mathbf{x}_n]$ descends to a \mathbb{F} -basis for R_n .

Applying Theorem 4.3 to Huang's basis of R_n , we get a collection of polynomials $\mathcal{C}_{n,k,r}$ given by

$$(4.25) \quad \mathcal{C}_{n,k,r} := \{e_\lambda(\mathbf{x}_n) \cdot \bar{\sigma}_\pi \cdot \mathbf{x}_{\text{Des}(\pi)} : \pi \in S_n \text{ and } \lambda \subseteq (n-r) \times k\}$$

which descends to a basis of $R_{n,k,r}$. The symmetric polynomial $e_\lambda(\mathbf{x}_n)$ has degree $|\lambda|$ and the symmetry of $e_\lambda(\mathbf{x}_n)$ gives

$$(4.26) \quad \sigma_i \cdot (e_\lambda(\mathbf{x}_n) \cdot \bar{\sigma}_\pi \cdot \mathbf{x}_{\text{Des}(\pi)}) = e_\lambda(\mathbf{x}_n) \sigma_i \cdot (\bar{\sigma}_\pi \cdot \mathbf{x}_{\text{Des}(\pi)}).$$

It follows that, for $\lambda \subseteq (n-r) \times k$ fixed, the collection of polynomials

$$(4.27) \quad \mathcal{C}_{n,k,r}(\lambda) := \{e_\lambda(\mathbf{x}_n) \cdot \bar{\sigma}_\pi \cdot \mathbf{x}_{\text{Des}(\pi)} : \pi \in S_n\}$$

descends inside $R_{n,k,r}$ to a \mathbb{F} -basis of a copy of R_n with degree shifted up by $|\lambda|$. \square

It may be tempting to try to prove Theorem 4.4 in the same fashion as Theorem 4.2 – by applying the short exact sequence of Lemma 4.1 directly and without appealing to Theorem 4.3. However, although the maps in this sequence commute with the action of $H_n(0)$, since $H_n(0)$ is not semisimple it is not *a priori* clear that this sequence splits in the category of $H_n(0)$ -modules. Theorem 4.4 guarantees that this sequence splits; the authors do not know of a more direct way to see this splitting.

Corollary 4.5. *Let n, k , and r be nonnegative integers with $r \leq n$.*

(1) *The length-degree bigraded quasisymmetric characteristic $\text{Ch}_{q,t}(R_{n,k,r})$ is given by*

$$(4.28) \quad \text{Ch}_{q,t}(R_{n,k,r}) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \text{Ch}_{q,t}(R_n) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \sum_{\pi \in S_n} q^{\text{maj}(\pi)} t^{\text{inv}(\pi)} F_{\text{Des}(\pi^{-1}),n},$$

where $F_{\text{Des}(\pi^{-1}),n}$ is the fundamental quasisymmetric function.

(2) *The degree graded quasisymmetric characteristic $\text{Ch}_q(R_{n,k,r})$ is in fact symmetric and given by*

$$(4.29) \quad \text{Ch}_q(R_{n,k,r}) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \text{Ch}_q(R_n) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \sum_{T \in \text{SYT}(n)} q^{\text{maj}(T)} s_{\text{shape}(T)}.$$

- (3) The $H_n(0)$ -module $R_{n,k,r}$ is projective. Its degree graded noncommutative characteristic $\mathbf{ch}_q(R_{n,k,r})$ is

$$(4.30) \quad \mathbf{ch}_q(R_{n,k,r}) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \mathbf{ch}_q(R_n) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \sum_{\alpha} q^{\text{maj}(\alpha)} \mathbf{s}_{\alpha},$$

where α ranges over all strong compositions of n , the major index is $\text{maj}(\alpha) = (\alpha_1) + (\alpha_1 + \alpha_2) + \dots$, and \mathbf{s}_{α} is the noncommutative ribbon Schur function.

Proof. Parts 1 and 2 follow from the work of Huang [12, Cor. 4.9] and Theorem 4.4. Since R_n is a projective $H_n(0)$ -module (see [12, Thm. 4.5]) and direct sums of projective modules are projective, we can apply [12, Cor. 8.4, $\mu = (1^n)$] to get Part 3. \square

Since the characteristics $\text{Ch}_{q,t}$ and Ch_t are defined in terms of the Grothendieck group $G_0(H_n(0))$ of $H_n(0)$, we may apply the short exact sequence of Lemma 4.1 to obtain Parts 1 and 2 of Corollary 4.5 more directly. However, since extensions of projective modules are not in general projective, Lemma 4.1 does not immediately imply that $R_{n,k,r}$ is a projective $H_n(0)$ -module.

Although Theorem 4.3 gives a collection of *polynomials* in $\mathbb{F}[\mathbf{x}_n]$ generalizing the GS monomials which descend to a basis of $R_{n,k,r}$, the authors have been unable to find a collection of *monomials* in $\mathbb{F}[\mathbf{x}_n]$ which generalizes the GS monomials and descends to a basis of $R_{n,k,r}$ (such monomial bases were found for the quotients appearing in the work of Haglund-Rhoades-Shimozono and Huang-Rhoades [10, 13]). Judging from the construction in [10, Sec. 5] and the Hilbert series of $R_{n,k,r}$, one might expect that the set of monomials

$$(4.31) \quad \{gs_{\pi} \cdot x_{\pi_1}^{i_1} \cdots x_{\pi_{n-r}}^{i_{n-r}} : \pi \in S_n \text{ and } k \geq i_1 \geq \cdots \geq i_{n-r} \geq 0\}$$

would descend to a basis of $R_{n,k,r}$, but this set of monomials is linearly dependent in the quotient in general. A potential combinatorial obstruction to finding a GS monomial basis for $R_{n,k,r}$ is the fact that the statistics inv and maj do *not* share the same distribution on $S_{n,k,r}$.

5. OPEN PROBLEMS

5.1. Bivariate generalization for $r = 1$. We propose a relationship between our quotient ring $R_{n,k,r}$ and the theory of Macdonald polynomials. In particular, consider the ideal $I'_{n,k,r} \subseteq \mathbb{Q}[\mathbf{x}_n]$ given by

$$(5.1) \quad I'_{n,k,r} := \langle p_{k+1}(\mathbf{x}_n), p_{k+2}(\mathbf{x}_n), \dots, p_{k+n}(\mathbf{x}_n), e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-r+1}(\mathbf{x}_n) \rangle$$

and let $R'_{n,k,r} := \mathbb{Q}[\mathbf{x}_n]/I'_{n,k,r}$ be the corresponding quotient. The ideal $I'_{n,k,r}$ is obtained from the ideal $I_{n,k,r}$ by replacing the homogeneous symmetric functions with power sum symmetric functions.

As with the quotient $R_{n,k,r}$, the quotient $R'_{n,k,r}$ has the structure of a graded S_n -module. Although the ideals $I_{n,k,r}$ and $I'_{n,k,r}$ are not equal in general, we present

Conjecture 5.1. *There is an isomorphism of graded S_n -modules $R_{n,k,r} \cong R'_{n,k,r}$.*

The main reason for preferring the quotient rings $R'_{n,k,r}$ over the quotient rings $R_{n,k,r}$ is that they generalize more readily to two sets of variables. Let $\mathbf{x}_n = (x_1, \dots, x_n)$ and $\mathbf{y}_n = (y_1, \dots, y_n)$ be two sets of n variables and let $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]$ be the polynomial ring in these variables. The symmetric group S_n acts on $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]$ by the *diagonal action* $\pi.x_i = x_{\pi_i}, \pi.y_i = y_{\pi_i}$.

For any $a, b \geq 0$, let $p_{a,b}(\mathbf{x}_n, \mathbf{y}_n)$ be the *polarized power sum*

$$(5.2) \quad p_{a,b}(\mathbf{x}_n, \mathbf{y}_n) := \sum_{i=1}^n x_i^a y_i^b.$$

Moreover, let \mathcal{M}_n be the set of the 2^n monomials $z_1 \dots z_n$ in $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]$ where $z_i \in \{x_i, y_i\}$ for all $1 \leq i \leq n$. For example, we have

$$\mathcal{M}_2 = \{x_1 x_2, x_1 y_2, y_1 x_2, y_1 y_2\}.$$

For a nonnegative integer k , let $DI_{n,k} \subseteq \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]$ be the ideal generated by the polarized power sums $p_{a,b}(\mathbf{x}_n, \mathbf{y}_n)$ with $a + b \geq k + 1$ together with the monomials in \mathcal{M}_n . Let $DR_{n,k} := \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/DI_{n,k}$ be the corresponding quotient, which is a bigraded S_n -module.

Conjecture 5.2. *The bigraded Frobenius image of $DR_{n,k}$ is given by the delta operator image*

$$\text{grFrob}(DR_{n,k}; q, t) = \Delta_{h_k e_n} e_n = \Delta_{s_{k+1, 1^{n-1}}} e_n = \Delta_{h_k} \nabla e_n.$$

The latter three quantities in the conjecture are trivially equal by the definition of the delta operator. When $k = 0$, the ring $DR_{n,0}$ is the classical diagonal coinvariant ring DR_n , so that Conjecture 5.2 reduces to Haiman's celebrated result [11] that $\text{grFrob}(DR_n) = \Delta_{e_n} e_n$. Setting the \mathbf{y}_n variables equal to zero in the quotient $DR_{n,k}$ yields the ring $R'_{n,k,1}$, so that the ring $R'_{n,k,1}$ conjecturally gives the analog of the coinvariant ring (for one set of variables) attached to the operator $\Delta_{h_k e_n}$.

The following proposition states that our module $R_{n,k,1}$ has graded Frobenius series which agrees with any of the delta operator expressions in Conjecture 5.2 upon setting $q = 0$ and $t = q$.

Proposition 5.3. *We have*

$$\text{grFrob}(R_{n,k,1}; t) = \Delta_{h_k e_n} e_n \big|_{q=0} = \Delta_{s_{k-1, 1^{n-1}}} e_n \big|_{q=0} = \Delta_{h_k} \nabla e_n \big|_{q=0}.$$

Proof. In this proof we will use the notation of plethysm; we refer the reader to [8] for the relevant details on plethysm and symmetric functions.

Let rev_t be the operator which reverses the coefficient sequences of polynomials with respect to the variable t . For a partition $\lambda \vdash n$, let $Q'_\lambda = Q'_\lambda(\mathbf{x}; t)$ be the corresponding Hall-Littlewood symmetric function. It is well known that the modified Macdonald polynomial $\tilde{H}_\lambda = \tilde{H}_\lambda(\mathbf{x}; q, t)$ satisfies

$$(5.3) \quad \tilde{H}_\lambda \big|_{q=0} = \text{rev}_t(Q'_\lambda).$$

This means that, for any symmetric function f and any partition $\lambda \vdash n$, we have

$$(5.4) \quad \Delta_f(\tilde{H}_\lambda) \big|_{q=0} = f(1, t, t^2, \dots, t^{\ell(\lambda)-1}) \cdot \text{rev}_t(Q'_\lambda),$$

where $\ell(\lambda)$ is the number of parts of λ .

In order to exploit Equation 5.4, we need to express e_n in terms of the modified Macdonald basis. This expansion is found in [8, Eqn. 2.72]: we have

$$(5.5) \quad e_n = \sum_{\lambda \vdash n} \frac{MB_\lambda \Pi_\lambda \tilde{H}_\lambda}{w_\lambda},$$

where

- $M = (1 - q)(1 - t)$,
- $B_\lambda = \sum_{c=(i,j) \in \lambda} q^{i-1} t^{j-i}$, where the sum is over all cells c with matrix coordinates (i, j) in the Ferrers diagram of λ ,
- $\Pi_\lambda = \prod_{c=(i,j) \neq (0,0)} (1 - q^{i-1} t^{j-1})$, where $c = (i, j)$ is a cell in λ other than the corner $(0, 0)$,
- $w_\lambda = \prod_{c \in \lambda} (q^{a(c)} - t^{l(c)+1})(t^{l(c)} - q^{a(c)+1})$, where the product is over all cells c in the Ferrers diagram of λ and $a(c), l(c)$ denote the arm and leg lengths of λ at c .

We apply the operator $\Delta_{h_k e_n} = \Delta_{h_k} \Delta_{e_n}$ to both sides of Equation 5.5 to get

$$(5.6) \quad \Delta_{h_k e_n} e_n = \sum_{\lambda \vdash n} h_k[B_\lambda] e_n[B_\lambda] \frac{MB_\lambda \Pi_\lambda \tilde{H}_\lambda}{w_\lambda}.$$

Setting $q = 0$ on both sides of Equation 5.6 gives

$$(5.7) \quad \Delta_{h_k e_n} e_n \big|_{q=0} = \left[\sum_{\lambda \vdash n} h_k[B_\lambda] e_n[B_\lambda] \frac{MB_\lambda \Pi_\lambda \tilde{H}_\lambda}{w_\lambda} \right]_{q=0}.$$

For any $\lambda \vdash n$ and any symmetric function f , we have $f[B_\lambda] \big|_{q=0} = f(1, t, t^2, \dots, t^{\ell(\lambda)-1})$. In particular, we have $e_n[B_\lambda] = 0$ unless $\lambda = (1^n)$ and Equation 5.7 reduces to

$$(5.8) \quad \Delta_{h_k e_n} e_n \big|_{q=0} = h_k(1, t, \dots, t^{n-1}) \cdot e_n(1, t, \dots, t^{n-1}) \cdot \left[\frac{MB_{(1^n)} \Pi_{(1^n)} \tilde{H}_{(1^n)}}{w_{(1^n)}} \right]_{q=0}.$$

The right hand side of Equation 5.8 simplifies to

$$(5.9) \quad \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_t \cdot \text{rev}_t(Q'_{(1^n)}) = \text{grFrob}(R_{n,k,1}; t)$$

where we used Theorem 4.2 at $r = 1$ and the well known fact that the graded Frobenius image of the classical coinvariant algebra R_n is $\text{grFrob}(R_n; t) = \text{rev}_t(Q'_{(1^n)})$. \square

5.2. Other bivariate generalizations. One may wonder if there is a bivariate generalization of the entire ring $R_{n,k,r}$, as we have only discussed the $r = 1$ case so far. While we have not been able to find a full generalization, there is some progress in the Hilbert series case. The *skewing operator* acts on a symmetric function f of degree d uniquely so that

$$(5.10) \quad \langle \partial f, g \rangle = \langle f, p_1 g \rangle$$

for all symmetric functions g of degree $d-1$, where the inner product is the usual Hall inner product on symmetric functions.

Given a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ of n positive integers, an α -*Tesler matrix* $U = (u_{i,j})_{1 \leq i,j \leq n}$ is an $n \times n$ upper triangular matrix with nonnegative integer entries such that, for $i = 1$ to n ,

$$(5.11) \quad u_{i,i} + u_{i,i+1} + \dots + u_{i,n} - (u_{1,i} + u_{2,i} + \dots + u_{i-1,i}) = \alpha_i.$$

We write $U \in \mathcal{T}(\alpha)$. For example, the matrix

$$U = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

satisfies $U \in \mathcal{T}(3, 2, 2, 3)$. The *weight* of an $n \times n$ α -Tesler matrix U is equal to

$$(5.12) \quad \text{wt}(U; q, t) = (-(1-q)(1-t))^{\text{pos}(U)-n} \prod_{u_{i,j} > 0} [u_{i,j}]_{q,t}$$

where $\text{pos}(U)$ is the number of positive entries in U and $[k]_{q,t}$ is the usual q, t -integer, i.e. $[k]_{q,t} = \frac{q^k - t^k}{q - t}$. For example, if U is the Tesler matrix shown above, we have $\text{pos}(U) = 7$ and

$$\text{wt}(U; q, t) = (-(1-q)(1-t))^{7-4} [2]_{q,t}^2 [3]_{q,t} [6]_{q,t}.$$

Finally, the α -*Tesler polynomial* is

$$(5.13) \quad \text{Tes}(\alpha; q, t) = \sum_{U \in \mathcal{T}(\alpha)} \text{wt}(U; q, t).$$

This corollary follows from work in [1, 9, 14].

Corollary 5.4.

$$(5.14) \quad \text{Hilb}(R_{n,k,r}; q) = \left[\begin{matrix} n+k-r \\ k \end{matrix} \right]_q \cdot [n]!_q$$

$$(5.15) \quad = \partial^{n-r+1} \Delta_{h_k} \partial^{r-1} \nabla e_n|_{t=0}$$

$$(5.16) \quad = \sum_{\substack{\alpha \models n+k \\ \ell(\alpha)=n \\ \alpha_1=\dots=\alpha_r=1}} \text{Tes}(\alpha; q, 0)$$

It would be interesting to find an extension of this corollary to the entire graded Frobenius series of $R_{n,k,r}$ for general r .

5.3. A Schubert basis. There is also a basis for $R_{n,k,r}$ given by certain Schubert polynomials. We let $\Pi_{n,k,r}$ be all the permutations π of $\{1, 2, \dots, n+k\}$ that satisfy

- all descents in π occur weakly left of position n , and
- $1, 2, \dots, r$ all appear in $\pi_1 \pi_2 \dots \pi_n$.

If $\pi \in \Pi_{n,k,r}$ is a permutation, let $\mathfrak{S}_\pi(\mathbf{x}_n)$ be the Schubert polynomial attached to π . Note that, since each π has no descents after position n , there are at most n variables that appear in the Schubert polynomial associated to π , so we have not truncated the variable set in any meaningful way. We will show that $\{\mathfrak{S}_\pi(\mathbf{x}_n^*) : \pi \in \Pi_{n,k,r}\}$ is a basis for $R_{n,k,r}$, where the asterisk represents the reversal of the vector of variables. This will follow from the fact that the leading terms are all (n, k, r) -good monomials.

Proposition 5.5. *Let $<$ be the lexicographic monomial order and let*

$$\mathcal{LT}_{n,k,r} = \{\text{in}_{<}(\mathfrak{S}_\pi(\mathbf{x}_n^*)) : \pi \in \Pi_{n,k,r}\}.$$

Then $\mathcal{LT}_{n,k,r} = \mathcal{M}_{n,k,r}$.

Proof. We will construct a bijection $\Phi : \Pi_{n,k,r} \rightarrow \mathcal{M}_{n,k,r}$ that satisfies $\Phi(\pi) = \text{in}_{<}(\mathfrak{S}_\pi(\mathbf{x}_n^*))$. The bijection itself is

$$(5.17) \quad \Phi(\pi) = \prod_{i=1}^n x_{n-i+1}^{d_i}$$

where d_i counts the number of $j > i$ such that $\pi_i > \pi_j$. The fact that $\Phi(\pi) = \text{in}_{<}(\mathfrak{S}_\pi(\mathbf{x}_n^*))$ follows directly from the definition of the Schubert polynomial. We need to show that $m = \Phi(\pi) \in \mathcal{M}_{n,k,r}$ and to construct its inverse. Our proof will be similar to that of Lemma 3.5. First, we check that

- we have $\mathbf{x}(S) \nmid m$ for all $S \subseteq [n]$ with $|S| = n - r + 1$, and
- we have $x_i^{k+i} \nmid m$ for all $1 \leq i \leq n$.

To check the first condition, we recall that $\mathbf{x}(S) = x_{s_1}^{s_1} x_{s_2}^{s_2-1} \dots x_{s_{n-r+1}}^{s_{n-r+1}-n+r}$ if $S = \{s_1 < s_2 < \dots < s_{n-r+1}\}$. Since $S \subseteq [n]$ and the entries 1 through r all appear in π_1 through π_n , there is some s_i such that $\pi_{s_i} \leq r$. Choose i as large as possible such that $\pi_{s_i} \leq r$. Since $j > n$ implies $\pi_j > r$, π_{s_i} can only be greater than at most $n - s_i$ entries to its right, i.e. $d_{s_i} \leq n - s_i$. Hence the power of x_{n-s_i+1} in m is at most $n - s_i$, which means $\mathbf{x}(S) \nmid m$. The second condition follows from the definition of m .

Given a monomial $m \in \mathcal{M}_{n,k,r}$, we would like to construct $\Phi^{-1}(m)$. This can be done using the usual bijection from codes (d_1, d_2, \dots, d_n) to permutations. For $i = 1$ to n , we choose π_i such that it is greater than exactly d_i of the entries in $[n+k]$ that have not already been placed to the left of position i in π . The second condition for (n, k, r) -good monomials implies that the result is an honest permutation, and the first condition implies that $1, 2, \dots, r$ all appear in the first n entries. \square

Corollary 5.6. $\{\mathfrak{S}_\pi(\mathbf{x}_n^*) : \pi \in \Pi_{n,k,r}\}$ descends to a basis for $R_{n,k,r}$.

It would be interesting to explore if this Schubert basis maintains many of the properties of the Schubert basis for the usual ring of coinvariants. For example, the following suggests that the structure constants of this Schubert basis are positive modulo $R_{n,k,r}$.

Question 5.7. For two permutations $\pi, \pi' \in \Pi_{n,k,r}$, is it always true that the product

$$(5.18) \quad \mathfrak{S}_\pi(\mathbf{x}_n^*) \times \mathfrak{S}_{\pi'}(\mathbf{x}_n^*)$$

has positive integer coefficients when expanded in the basis $\{\mathfrak{S}_\pi(\mathbf{x}_n^*) : \pi \in \Pi_{n,k,r}\}$ modulo $I_{n,k,r}$? Using SAGE, we have checked that this is true for $1 \leq n, k \leq 4$ and $0 \leq r \leq n$. If so, do these coefficients count intersections in some family of varieties?

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